Computing the Numerical Scale of the Linguistic Term Set for the 2-Tuple Fuzzy Linguistic Representation Model

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Abstract—When using linguistic approaches to solve decision problems, we need the techniques for computing with words (CW). Together with the 2-tuple fuzzy linguistic representation models (i.e., the Herrera and Martínez model and the Wang and Hao model), some computational techniques for CW are also developed. In this paper, we define the concept of numerical scale and extend the 2-tuple fuzzy linguistic representation models under the numerical scale. We find that the key of computational techniques based on linguistic 2-tuples is to set suitable numerical scale with the purpose of making transformations between linguistic 2-tuples and numerical values. By defining the concept of the transitive calibration matrix and its consistent index, this paper develops an optimization model to compute the numerical scale of the linguistic term set. The desired properties of the optimization model are also presented. Furthermore, we discuss how to construct the transitive calibration matrix for decision problems using linguistic preference relations and analyze the linkage between the consistent index of the transitive calibration matrix and one of the linguistic preference relations. The results in this paper are pretty helpful to complete the fuzzy 2-tuple representation models for CW.

Index Terms—Computing with words (CW), linguistic modeling, linguistic variables, numerical scale, transitive calibration.

I. INTRODUCTION

The linguistic approach deals with qualitative aspects that are represented in qualitative terms by means of linguistic variables. It has received increasing attention since its appearance, and been used successfully in a wide range of applications, such as information retrieval [5], [27], [28], [34], and [35], decision-making [2], [6], [7], [11], [14], [15], [17], [30], [33], and [43], consensus process [3], [4], and [29], and aggregation operators [12], [13], [18], [21], [39], [41], [44]–[48], and [52]. When using linguistic approaches to solve decision problems, we need the techniques for computing with words (CW). Herrera and Martínez [22] surveyed that there are three linguistic computational models:

1) the semantic model [11], [36]–[38], [49]–[52];
2) the symbolic model [12], [21];
3) the model based on linguistic 2-tuples [22], [43].

The semantic model is based on the extension principle, and makes operations on the fuzzy numbers that support the semantics of the linguistic terms. The symbolic model makes computations on the indexes of the linguistic terms. Herrera and Martínez [22] pointed out that the results of these two models usually do not match with any of the initial linguistic terms; therefore, an approximation process must be developed to express the result in the initial expression domain.

Herrera and Martínez [22] proposed the notable 2-tuple fuzzy linguistic representation model and defined the corresponding computational technique to deal with the 2-tuples. Using the 2-tuple linguistic representation model, the Herreras and Martínez [24] further developed a linguistic decision model dealing with multigranular linguistic contexts in the linguistic hierarchy structure. Although the Herrera and Martínez model has no loss of information (in a sense) when one applies it in a computational stage for CW, Herrera and Martínez [22, Conclusion] have pointed out that this approach is only suitable for linguistic term sets that are uniformly and symmetrically distributed.

Wang and Hao [42] extended the Herrera and Martínez model, providing an interesting generalized version (i.e., the proportional 2-tuple fuzzy linguistic representation model) for CW, based on the concepts of “symbolic proportion and the canonical characteristic values (CCVs) of linguistic terms. Wang and Hao’s model can deal with linguistic variables with linguistic term sets that are not uniformly and symmetrically distributed. However, the semantic of linguistic terms in the linguistic term set used in the Wang and Hao model can only be defined for symmetrical trapezoidal fuzzy numbers. Thus, Wang and Hao [42, Conclusion] suggested on investigating how to generalize their proposal by considering linguistic 2-tuples under more general contexts.

In this paper, we present the concept of the numerical scale for the linguistic term set and argue that the key of the computational techniques based on 2-tuple fuzzy linguistic representation models is to set suitable numerical scale with the purpose of making transformations between linguistic 2-tuples and numerical values. The main aim of this paper is to further complete the 2-tuple fuzzy linguistic representation models for CW by providing an optimization model to compute the numerical scale of the linguistic term set. The rest of this paper is organized as follows. In Section II, we shall present a brief review of two 2-tuple fuzzy linguistic representation models
(i.e., the Herrera and Martínez model and the Wang and Hao model). In Section III, we extend the existing 2-tuple fuzzy linguistic representation models under the concept of the numerical scale. In Section IV, we present an optimization model to compute the numerical scale of the linguistic term set by developing the concept of the transitive calibration matrix. In Section V, we discuss how to construct the transitive calibration matrix from linguistic preference relations provided by decision makers and show a good linkage between the consistent index of the transitive calibration matrix and the one of the linguistic preference relation. In Section VI, two illustrative examples are provided, and finally, some concluding remarks are included in Section VII.

II. REVIEW OF 2-TUPLE FUZZY LINGUISTIC REPRESENTATION MODELS

The basic notations and operational laws of linguistic variables are introduced in [11], [12], [22], [42], and [46]. Let \( S = \{s_0 | \alpha = 0, 1, 2, \ldots, g\} \) be a linguistic term set with odd cardinality. The term \( s_0 \) represents a possible value for a linguistic variable, and it is required that the linguistic term set should satisfy the following characteristics.

1) The set is ordered: \( s_0 > s_1 \), if and only if \( \alpha > \beta \).
2) There is a negation operator.

Note 1: The negation is usually used to change a positive number to a negative number, and an antonym is used to change a large number to a small number. We consider that antonym is needed in linguistic term sets. However, since the name negation has been used in 2-tuple fuzzy linguistic representation models for a long time, we also use the name negation in this paper.

We also call this linguistic term set \( S \) as the linguistic scale. For example, \( S \) can be defined as

\[
S = \{s_0 = N\text{(none)}, s_1 = V\text{L}\text{(very low)} \}
\]

\[
s_2 = L\text{(low)}, s_3 = M\text{(medium)}, s_4 = H\text{(high)} \}
\]

\[
s_5 = VH\text{(very high)}, s_6 = P\text{(perfect)} \}
\]

The semantics of the elements in a linguistic term set is given by fuzzy numbers (defined in the \([0, 1]\) interval), which are described by membership functions. In general, we can consider that membership functions are linear triangular membership functions or linear trapezoidal membership functions. For instance, the linear trapezoidal membership function is achieved by the 4-tuple \((a, b, c, d)\), \(b \) and \(c \) indicate the interval in which the membership value is 1, and \(a \) and \(d \) are the left and right limits of the definition domain of a trapezoidal membership function. Fuzzy numbers with trapezoidal membership functions are denoted by \( T[a, b, c, d] \). In a linguistic term set, the midterm represents an assessment of “approximately 0.5.” Similar to the paper presented in Herrera et al. [20], this paper also assumes that in the investigated linguistic term set, there exists a midterm. We also denote \( s^* \in S \) as the midterm of \( S \).

Let \( s \in S \), we denote \( p(s) \) as the position index of \( s \) and call it the gradation of \( s \) in \( S \). For example, if \( s = s_\alpha \), then \( p(s) = \alpha \).

A. Herrera and Martínez Model

Herrera and Martínez [22]–[24] contributed a 2-tuple fuzzy linguistic representation model. In [20], Herrera et al. presented a brief review of this 2-tuple fuzzy linguistic representation model.

Definition 1 [22]: Let \( \beta \in [0, g] \) be a number in the granularity interval of the linguistic term set \( S = \{s_0, \ldots, s_g\} \), and let \( i = \text{round}(\beta) \) and \( \alpha = \beta - i \) be two values such that \( i \in [0, g] \) and \( \alpha \in [-0.5, 0.5] \). Then, \( \alpha \) is called a symbolic translation, with round being the usual rounding operation.

The Herrera and Martínez model represents the linguistic information by means of 2-tuples \((s_i, \alpha_i)\), where \( s_i \in S \) and \( \alpha_i \in [-0.5, 0.5] \). This linguistic representation model defines a function with the purpose of making transformations between linguistic 2-tuples and numerical values.

Definition 2 [22]: Let \( S = \{s_0, \ldots, s_g\} \) be a linguistic term set and \( \beta \in [0, g] \) a value representing the result of a symbolic aggregation operation; then, the 2-tuple that expresses the equivalent information to \( \beta \) is obtained with the following function:

\[
\Delta : [0, g] \to S \times [-0.5, 0.5]
\]

\[
\Delta(\beta) = (s_i, \alpha) \text{ with } \alpha = \beta - i, \alpha \in [-0.5, 0.5].
\]

Clearly, \( \Delta \) is one to one. For convenience, its range is denoted as \( S \). Then, \( \Delta \) has an inverse function with \( \Delta^{-1} : S \to [0, g] \) with \( \Delta^{-1}(s_i, \alpha) = i + \alpha \).

A computational model has been developed for the Herrera and Martínez model in which there exist the following.

1) A 2-tuple comparison operator: Let \((s_k, \alpha_1)\) and \((s_l, \alpha_2)\) be two 2-tuples. Then:

- a) if \( k < l \), then \((s_k, \alpha_1)\) is smaller than \((s_l, \alpha_2)\);
- b) if \( k = l \), then

  i) \( \alpha_1 = \alpha_2 \), then \((s_k, \alpha_1), (s_l, \alpha_2)\) represents the same information;
  ii) \( \alpha_1 < \alpha_2 \), then \((s_k, \alpha_1)\) is smaller than \((s_l, \alpha_2)\).

2) A 2-tuple negation operator

\[
\text{Neg}((s_i, \alpha_i)) = \Delta(g - (\Delta^{-1}(s_i, \alpha_i))).
\]

3) Some 2-tuple aggregation operators have been developed, such as the linguistic ordered weighted aggregation (LOWA) operator, the weighted average operator, the OWA operator, etc. (see [21] and [22]).

B. Wang and Hao Model

Wang and Hao [42] provided a new (proportional) 2-tuple fuzzy linguistic representation model for CW. This model represents the linguistic information by means of proportional 2-tuples (see Definition 3).

Definition 3 [42]: Let \( S = \{s_0, s_1, \ldots, s_g\} \) be a linguistic term set \( I = [0, 1] \) and

\[
IS \equiv I \times S = \{(\alpha, s_i) | \alpha \in [0, 1] \text{ and } i = 0, 1, \ldots, g\}.
\]
Given a pair \((s_i, s_{i+1})\) of two successive terms of \(S\), any two elements \((\alpha, s_i), (\beta, s_{i+1})\) of \(IS\) is called a symbolic proportion pair and \(\alpha, \beta\) are called a pair of symbolic proportions of the pair \((s_i, s_{i+1})\), if \(\alpha + \beta = 1\). A symbolic proportion pair \((\alpha, s_i), (1 - \alpha, s_{i+1})\) will be denoted \((\alpha, 1 - \alpha)_{s_i, s_{i+1}}\), and the set, of all the symbolic proportion pairs is denoted \(\mathfrak{S}\), i.e., \(\mathfrak{S} = \{(\alpha, 1 - \alpha)_{s_i, s_{i+1}}|\alpha \in [0, 1] \text{ and } i = 0, 1, \ldots, g - 1\}\).

In the Wang and Hao model, the set \(\mathfrak{S}\) is called the proportional 2-tuple set generated by \(S\), and the members of \(\mathfrak{S}\) are called linguistic proportional 2-tuples.

The semantic of linguistic terms used in Wang and Hao’s model is defined by symmetrical trapezoidal fuzzy numbers \(T[b - \sigma, b, c, c + \sigma]\), which are varying in a 1-D characteristic. If the semantic of \(s_i\) is defined by \(T[b_i - \sigma_i, b_i, c_i, c_i + \sigma_i]\), Wang and Hao argued that the CCV of \(s_i\) is \((b_i + c_i)/2\), i.e., \(CCV(s_i) = (b_i + c_i)/2\). Wang and Hao extended CCV to \(\mathfrak{S}\) as follows.

**Definition 4** [42]: Let \(S, \mathfrak{S}\), and CCV on \(S\) be as before. For \((\alpha, s_i), (1 - \alpha)_{s_i, s_{i+1}}\) \(\in \mathfrak{S}\), the function CCV on \(\mathfrak{S}\) is

\[
CCV(\alpha, s_i) = \alpha CCV(s_i) + (1 - \alpha)CCV(s_{i+1}).
\]

A computational model has been developed for the Wang and Hao model, in which there exist the following.

1) A 2-tuple comparison operator: Let \((\alpha, s_i), (1 - \alpha)_{s_i, s_{i+1}}\) and \((\beta, s_j), (1 - \beta)_{s_j, s_{j+1}}\) be two proportional 2-tuples. Then:
   a) if \(i < j\), then
      i) \((\alpha, s_i), (1 - \alpha)_{s_i, s_{i+1}}\), \((\beta, s_j), (1 - \beta)_{s_j, s_{j+1}}\) represent the same information when \(i = j = 1\) and \(\alpha = 0, \beta = 1\);
      ii) \((\alpha, s_i), (1 - \alpha)_{s_i, s_{i+1}}\) < \((\beta, s_j), (1 - \beta)_{s_j, s_{j+1}}\), otherwise;
   b) if \(i = j\), then
      i) if \(\alpha = \beta\), then \((\alpha, s_i), (1 - \alpha)_{s_i, s_{i+1}}\), \((\beta, s_j), (1 - \beta)_{s_j, s_{j+1}}\) represent the same information;
      ii) if \(\alpha < \beta\), then \((\alpha, s_i), (1 - \alpha)_{s_i, s_{i+1}}\) > \((\beta, s_j), (1 - \beta)_{s_j, s_{j+1}}\).

2) The negation operator over proportional 2-tuples

\[
\neg((\alpha, s_i), (1 - \alpha)_{s_i, s_{i+1}}) = ((1 - \alpha)_{s_{g-i+1}, s_{g-i}}).
\]

3) Based on CCVs, Wang and Hao developed some weighted and ordered weighted aggregation operators of proportional 2-tuples as well.

Wang and Hao [42] presented an interesting transform function between \(S\) and \(\mathfrak{S}\) (see Proposition 1).

**Proposition 1** [42]: Let \(S, \mathfrak{S}\), and \(\mathfrak{S}\) be as before. Define \(h: \mathfrak{S} \rightarrow \mathfrak{S}\) by

\[
h((\alpha, s_i), (1 - \alpha)_{s_i, s_{i+1}}) = \begin{cases} (s_i - 1, -\alpha), & 0 \leq \alpha \leq 1/2 \\
(s_i, 1 - \alpha), & 1/2 < \alpha \leq 1. \end{cases}
\]

Then, \(h\) is a bijection.

Let \(h^{-1}\) be the inverse function of \(h\). If \(x \in \mathfrak{S}\), and then we say \(h^{-1}(x)\) is the corresponding linguistic proportional 2-tuple in \(\mathfrak{S}\).

### III. Extension of 2-Tuple Fuzzy Linguistic Representation Models

In Section II, we introduce two 2-tuple fuzzy linguistic representation models (i.e., the Herrera and Martinez model and the Wang and Hao model) and some corresponding computational techniques for CW. Though these two models are based on different formats of linguistic 2-tuples, we find that the key task of the 2-tuple fuzzy linguistic representation models is to define a function with the purpose of making transformations between linguistic 2-tuples and numerical values. In this section, we extend the 2-tuple fuzzy linguistic representation models by formally proposing the concept of numerical scale.

**Definition 5**: Let \(S = \{s_i|\alpha = 0, 1, 2, \ldots, g\}\) be a linguistic term set and \(R\) be real number set. We define the function \(NS: S \rightarrow R\) as a numerical scale of \(S\) and call \(NS(s_i)\) the numerical index of \(s_i\).

**Definition 6**: Let \(S, \mathfrak{S}\), and \(NS\) on \(S\) be as before. For \((s_i, x) \in \mathfrak{S}\), we define the numerical scale \(NS\) on \(\mathfrak{S}\) by

\[
\neg NS((s_i, x)) = \begin{cases} NS(s_i) + x \times (NS(s_{i+1}) - NS(s_i)), & x \geq 0 \\
NS(s_i) + x \times (NS(s_{i+1}) - NS(s_{i-1})), & x < 0. \end{cases}
\]

For notational simplicity, \(\neg NS\) will also be denoted by \(NS\) in this paper.

**Definition 7**: If \(NS(s_i) < NS(s_{i+1})\), for \(i = 0, 1, \ldots, g - 1\), we say the numerical scale \(NS\) on \(S\) is ordered. If \(NS(s_i) \leq NS(s_{i+1})\), for \(i = 0, 1, \ldots, g - 1\), we say the numerical scale \(NS\) on \(S\) is weak ordered. Let \(x, y \in \mathfrak{S}\) and \(x < y\). If \(NS(x) < NS(y)\), we say the numerical scale \(NS\) on \(\mathfrak{S}\) is ordered. If \(NS(x) \leq NS(y)\), we say the numerical scale \(NS\) on \(\mathfrak{S}\) is weak ordered.

**Proposition 2**: If \(NS\) on \(S\) is ordered, then \(NS\) on \(\mathfrak{S}\) is ordered.

**Proof**: Let \((s_k, \alpha_1), (s_l, \alpha_2) \in \mathfrak{S}\) and \((s_k, \alpha_1) < (s_l, \alpha_2)\). Then, we consider the following two cases:

- **Case A**: \(k = l\) and \(\alpha_1 < \alpha_2\). In this case, we further consider the following three subcases:
  - **Subcase A1**: \(\alpha_1 \geq 0, \alpha_2 \geq 0\). In this subcase, according to Definitions 6 and 7, we have
    \[
    NS((s_k, \alpha_1)) = NS(s_k) + \alpha_1 \times (NS(s_{k+1}) - NS(s_k)) < NS(s_l) + \alpha_2 \times (NS(s_{l+1}) - NS(s_l)) = NS((s_l, \alpha_2)).
    \]
  - **Subcase A2**: \(\alpha_1 < 0, \alpha_2 < 0\). In this subcase, according to Definitions 6 and 7, we have
    \[
    NS((s_k, \alpha_1)) = NS(s_k) + \alpha_1 \times (NS(s_{k}) - NS(s_{k-1})) < NS(s_l) + \alpha_2 \times (NS(s_{l}) - NS(s_{l-1})) = NS((s_l, \alpha_2)).
    \]
Subcase A3: $\alpha_1 < 0, \alpha_2 \geq 0$. In this subcase

$$NS((s_k, \alpha_1)) = NS(s_k) + \alpha_1 \times (NS(s_k) - NS(s_{k-1}))$$

$$\leq NS(s_{k+1}) - NS(s_k)$$

$$= NS((s_{k+1}), \alpha_2)).$$

Consequently, $NS((s_k, \alpha_1)) < NS((s_{k+1}), \alpha_2)).$

Case B: $k < l$. In this case, from the proof of Case A, we have $NS(s_{k}, \alpha_1) < \lim_{\alpha_1 \to 0} NS(s_k, \alpha_1)$ and $NS(s_l, -0.5) \leq NS(s_l, -0.5)$. Therefore

$$NS((s_k, \alpha_1)) < NS(s_k) + 0.5 \times (NS(s_{k+1}) - NS(s_k))$$

$$= NS(s_{k+1}) - 0.5 \times (NS(s_{k+1}) - NS(s_k))$$

$$= NS((s_{k+1}), -0.5))$$

$$\leq NS(s_{k+1}) + NS(s_k) - NS(s_l)$$

$$= NS(s_l) + NS(s_{l-1})$$

$$= NS((s_l, -0.5)) \leq NS((s_l, -0.5)).$$

Consequently, $NS((s_l, -0.5)) < NS((s_l, -0.5))$, which means $NS$ on $\mathbb{S}$ is ordered. This completes the proof of Proposition 2.

Similar to Proposition 2, we can also prove that if $NS$ on $S$ is weak ordered, and then, $NS$ on $\mathbb{S}$ is weak ordered.

From Proposition 2, we obtain Corollary 1 and Corollary 2.

Corollary 1: If $NS$ on $\mathbb{S}$ is ordered, then $NS$ is a bijection from $\mathbb{S}$ to $[NS(s_0), NS(s_n)]$.

Corollary 2: Let $NS$ be an ordered numerical scale on $\mathbb{S}$ and $x, y \in \mathbb{S}$. Then, (1) $x < y$, iff $NS(x) < NS(y)$, and (2) $x = y$, iff $NS(x) = NS(y)$.

Proposition 3: When setting $NS(s_i) = i$, for $i = 0, 1, \ldots, g$, we have $NS(s_0, x_a) = \Delta^{-1}(s_0, x_a)$, for any $(s_0, x_a) \in S$.

Proof: When setting $NS(s_i) = i$, for $i = 0, 1, \ldots, g$, we have $NS(s_0, x_a) = \alpha + x_a$. Since $\Delta^{-1}(s_0, x_a) = \alpha + x_a$, consequently, $NS((s_0, x_a)) = \Delta^{-1}(s_0, x_a)$, for any $(s_0, x_a) \in \mathbb{S}$.

Proposition 4: When setting $NS(s_i) = CCV(s_i)$, for $i = 1, 2, \ldots, g$, we have $NS(s_0, x_a) = CCV(h^{-1}(s_0, x_a))$, for any $(s_0, x_a) \in \mathbb{S}$.

Proof: Consider the following two cases:

Case A: $x_a \geq 0$. In this case, according to Proposition 1, we have $h^{-1}(s_0, x_a) = (1 - x_a)s_0, x_a, s_{a+1})$. From Definitions 4 and 6, we have

$$NS((s_0, x_a)) = (1 - x_a) \times NS(s_0) + x_a \times NS(s_{a+1})$$

$$= (1 - x_a) \times CCV(s_0) + x_a \times CCV(s_{a+1})$$

$$= CCV((1 - x_a)s_0, x_a, s_{a+1}))$$

$$= CCV(h^{-1}(s_0, x_a)).$$

Case B: $x_a < 0$. In this case, $h^{-1}(s_0, x_a) = ((-x_a)s_{a-1}, 1 + x_a)s_0)$. From Definitions 4 and 6, we have

$$NS((s_0, x_a)) = (1 + x_a) \times NS(s_0) - x_a \times NS(s_{a-1})$$

$$= (1 + x_a) \times CCV(s_0) - x_a \times CCV(s_{a-1})$$

$$= CCV(((-x_a)s_{a-1}, (1 + x_a)s_0))$$

$$= CCV(h^{-1}((s_0, x_a))).$$

Proposition 3 shows that we can obtain the Herrera and Martínez model when setting $NS(s_i) = i$. Proposition 4 shows that we can obtain another 2-tuple fuzzy linguistic representation model when setting $NS(s_i) = CCV(s_i)$. This 2-tuple fuzzy linguistic representation model has similarity to the Wang and Hao model. The main difference between them lies in using different formats of linguistic 2-tuples.

Naturally, when setting different numerical scales, we can obtain different versions of 2-tuple fuzzy linguistic representation models. A natural question is how to set a suitable numerical scale $NS$ for the linguistic term set $S$. In particular, different decision makers who may have different background and knowledge often need to set different numerical scales.

IV. COMPUTING THE NUMERICAL SCALE OF THE LINGUISTIC TERM SET

A. Transitive Calibration Matrix

Let $S = \{s_0, s_1, \ldots, s_g\}$ be as before. Let $A_1, A_2$, and $A_3$ be three alternatives, and let $s_i, s_j \in S$ be two linguistic terms. We suppose that the decision maker knows that

1) the preference intensity between $A_1$ and $A_2$ is “$s\alpha$”;

2) the preference intensity between $A_2$ and $A_3$ is “$s\beta$”.

Then, let the decision maker answer the preference intensity between $A_1$ and $A_2$ using a linguistic term. We call this question to the decision maker as the transitive calibration between $s_i$ and $s_j$, and symbolize it as $s_i, s_j$. If the answer provided by the decision maker is the linguistic term $x_i$, we consider that the result of the transitive calibration between $s_i$ and $s_j$ is $x_i$, and symbolize it as $x = s_i, s_j$.

Definition 8: We say $E = (e_{ij}(g+1)\times(g+1))$, where $e_{ij} = s_{i-1}, s_{j-1}$ and $e_{ij} \in S$ is a discrete transitive calibration matrix of $S$.

Definition 9: We say $E = (e_{ij}(g+1)\times(g+1))$, where $e_{ij} = s_{i-1}, s_{j-1}$ and $e_{ij} \in S$ is a continuous transitive calibration matrix of $S$.

Definition 10: Let $E = (e_{ij}(g+1)\times(g+1))$ be a transitive calibration matrix of $S$. We say $E$ is symmetrical if $e_{ij} = e_{ji}$ for $e_{ij}, e_{ji} \neq 0$ and $i, j = 1, 2, \ldots, g + 1$.

Definition 11: Let $E = (e_{ij}(g+1)\times(g+1))$ be a transitive calibration matrix of $S$. We say $E$ is transitive if $e_{ij} \leq e_{ks}$ for $e_{ij}, e_{ks} \neq 0$, $i, k, s \leq 0$, and $j \leq 0$.

Remark 1: Generally, the transitive calibration matrices provided by decision makers are discrete, and the continuous transitive calibration matrices can appear in the construction process using the algorithm introduced in Section V-A. If some of the elements in $E$ cannot be given by the decision maker, which we denote by null, and the others satisfy $e_{ij} = s_{i-1}, s_{j-1}$, we say $E$ is an incomplete transitive calibration matrix of $S$. For notational simplicity, in this paper, incomplete transitive calibration matrix will also be called by transitive calibration matrix.

Remark 2: Different decision makers who may have different background and knowledge often provide different transitive calibration matrices for the same linguistic term set. In general,
we think that the transitive calibration matrices provided by rational decision makers should be symmetrical and transitive.

Furthermore, we define the consistent index of the transitive calibration matrix under numerical scale.

**Definition 12:** Let \( E = (e_{ij})_{(g+1) \times (g+1)} \) be a transitive calibration matrix of \( S \) and \( NS \) be a numerical scale on \( S \). We define the consistency index (CI) of \( E \) under \( NS \)

\[
CI(E, NS) = \frac{1}{m} \sum_{\beta=1, e_{ij} \neq \text{null} \atop \beta} \sum_{\alpha=1}^{g+1} \left| NS(s_{\alpha}) - NS(s_{\beta}) \right|
\]

where \( m \) is the number of the non-null elements of \( E \), and \( s^* \) is the midterm of \( S \).

**Proposition 5:** Let \( E = (e_{ij})_{(g+1) \times (g+1)} \) be a transitive calibration matrix of \( S \) and \( NS \) be an ordered numerical scale on \( S \). If \( CI(E, NS) = 0 \), then \( E \) is symmetrical and transitive.

**Proof:** Since \( CI(E, NS) = 0 \), we have

\[
NS(e_{ij}) = NS(s_{ij}) + NS(s_{ij}) - NS(s^*)
\]

and

\[
NS(e_{ij}) = NS(s_{ij}) + NS(s_{ij}) - NS(s^*)
\]

Thus, \( NS(e_{ij}) = NS(e_{ij}) \). Since \( NS \) is ordered, according to Corollary 1, we have \( e_{ij} = e_{ij} \). Therefore, \( E \) is symmetrical.

Moreover, when \( NS \) is ordered, we similarly have \( NS(e_{ij}) > NS(e_{ij}) \). Consequently, \( e_{ij} > e_{ij} \). Therefore, \( E \) is transitive.

This completes the proof of Proposition 5.

**Definition 13:** Let \( E = (e_{ij})_{(g+1) \times (g+1)} \) be a transitive calibration matrix of \( S \) and \( NS \) be the ordered numerical scale on \( S \). We say that \( E \) is consistent, if \( CI(E, NS) = 0 \).

From Proposition 5 and Definition 13, we have that the consistency of the transitive calibration matrix under the ordered numerical scale is a stronger condition than the symmetry and transitivity. In Section V-B, we will construct a good linkage of the consistency index of the transitive calibration matrix to the additive transitivity property introduced by Alonso et al. [1], Herrera-Viedma et al. [25], [26], and Tanino [40] and analyze the physical implication of this consistency index in detail.

**B. Computing the Numerical Scale From the Transitive Calibration Matrix**

As stated in earlier sections, when setting different numerical scales, we can obtain different versions of 2-tuple fuzzy linguistic representation models. In the existing studies, the decision maker using the 2-tuple fuzzy linguistic representation models can only set the numerical scale, according to his/her experience or the existing model (i.e., the Herrera and Martínez model and the Wang and Hao model). Thus, this section proposes an optimization model to help the decision maker set a suitable numerical scale \( NS \) of the linguistic term set \( S \).

If the decision maker provides initial numerical indexes \( b_\alpha \) for \( s_\alpha \), according to his/her experience or the existing model (i.e., the Herrera and Martínez model and the Wang and Hao model), then we hope that the deviation degree between \( NS(s_\alpha) \) and \( b_\alpha \) is minimal, namely

\[
\min_{NS} \frac{1}{g+1} \sum_{\alpha=0}^{g} \left| NS(s_\alpha) - b_\alpha \right|.
\]  

Moreover, we hope to find the numerical scale \( NS \) under which \( E \) has a good consistent index, namely

\[
\min_{NS} CI(E, NS)
\]

i.e.,

\[
\min_{NS} \left( \frac{1}{m} \sum_{\beta=1, e_{ij} \neq \text{null} \atop \beta} \sum_{\alpha=1}^{g+1} \left| NS(s_{\alpha}) - NS(s_{\beta}) \right| + NS(s_{\beta}) - NS(s_{\alpha}) - NS(s^*) \right)
\]

At the same time, the decision maker using the 2-tuple fuzzy linguistic representation models often demands that \( NS \) has a support \( \bar{NS}, NS \), that is

\[
NS \leq NS(s_\alpha) \leq \bar{NS}, \quad \alpha = 0, 1, \ldots, g.
\]

**Remark 3:** When the decision maker provides initial values \( b_\alpha, \alpha = 0, 1, \ldots, g \), we should suppose that \( NS \leq b_\alpha \leq b_\alpha + 1 \leq NS \). For the Herrera and Martínez model, \( \bar{NS} = 0 \), and \( NS = g \), where as \( \bar{NS} = 0 \) and \( NS = 1 \) in the Wang and Hao model.

In this way, we construct the following two-objective constrained optimization model:

\[
\begin{align*}
\min_{NS} & \left( z_1 = \frac{1}{m} \sum_{\beta=1, e_{ij} \neq \text{null} \atop \beta} \sum_{\alpha=1}^{g+1} \left| NS(s_{\alpha}) - NS(s_{\beta}) \right| + NS(s_{\beta}) - NS(s_{\alpha}) - NS(s^*) \right) \\
\text{s.t.} & \sum_{\alpha=1}^{g} |NS(s_\alpha) - b_\alpha| \leq \lambda z_2 \text{ for } \lambda \in [0, 1] \text{ and } z_2 = \frac{1}{g+1} \sum_{\alpha=1}^{g} |NS(s_\alpha) - b_\alpha| \leq \lambda z_1 \text{ for } \lambda \in [0, 1],
\end{align*}
\]

Using the linear weighted summation method [16], the model (5) can be transformed into a single objective optimization model, i.e.,

\[
\begin{align*}
\min_{NS} & \left( z = z_1 + (1-\lambda) z_2 \right) \\
\text{s.t.} & \sum_{\alpha=1}^{g} |NS(s_\alpha) - b_\alpha| \leq \bar{NS}, \quad \alpha = 0, 1, \ldots, g,
\end{align*}
\]

where \( 0 \leq \lambda \leq 1, \lambda = 1, \) and \( 1, \lambda \) denote the relative importance of objective \( z_1 \) and objective \( z_2 \), respectively.

We use four transformed decision variables: \( c_{\alpha, \beta} = NS(s_{\alpha-1}) + NS(s_{\beta-1}) - NS(s_{\alpha}) - NS(s_{\beta}) \), \( x_\alpha = NS(s_\alpha) - b_\alpha \), and \( y_\alpha = x_\alpha \). Then, we transform model (6) into the following linear programming model:

\[
\begin{align*}
\min_{NS} & \frac{\lambda}{m} \sum_{\beta=1, e_{ij} \neq \text{null} \atop \beta} \sum_{\alpha=1}^{g+1} d_{\alpha, \beta} + \frac{1-\lambda}{g+1} \sum_{\alpha=0}^{g} y_\alpha
\end{align*}
\]

subject to

\[
\bar{NS} \leq NS(s_\alpha) \leq \bar{NS}, \quad \alpha = 0, 1, \ldots, g
\]

\[
c_{\alpha, \beta} = NS(s_{\alpha-1}) + NS(s_{\beta-1}) - NS(s_{\alpha}) - NS(s_{\beta}) \]

\[
-NS(s^*), \quad \alpha, \beta = 1, 2, \ldots, g + 1
\]
\[ d_{\alpha,\beta} \geq c_{\alpha,\beta}, \quad \alpha, \beta = 1, 2, \ldots, g + 1 \]  
\[ d_{\alpha,\beta} \geq -c_{\alpha,\beta}, \quad \alpha, \beta = 1, 2, \ldots, g + 1 \]  
\[ x_\alpha = NS(s_\alpha - b_\alpha, \quad \alpha = 0, 1, \ldots, g \]  
\[ y_\alpha \geq x_\alpha, \quad \alpha = 0, 1, \ldots, g \]  
\[ y_\alpha \geq -x_\alpha, \quad \alpha = 0, 1, \ldots, g. \]  

Constraints (9)–(11) enforce that \( d_{\alpha,\beta} \geq |c_{\alpha,\beta}| = |NS(s_{\alpha-1}) + NS(s_{\beta-1}) - NS(e_{\alpha,\beta}) - NS(s^*)| \). According to the objective function [i.e., (7)], we find that any feasible solutions with \( d_{\alpha,\beta} > |c_{\alpha,\beta}| \) are not the optimal solution to model (7)–(14). Thus, constraints (9)–(11) can guarantee that \( d_{\alpha,\beta} = |c_{\alpha,\beta}| = |NS(s_{\alpha-1}) + NS(s_{\beta-1}) - NS(e_{\alpha,\beta}) - NS(s^*)| \) in model (7)–(14). Similarly, constraints (12)–(14) guarantee that \( y_\alpha = |x_\alpha| = |NS(s_\alpha) - b_\alpha| \) in model (7)–(14).

**Remark 4:** Clearly, \( NS(s_\alpha) = b_\alpha, \alpha = 0, 1, \ldots, g \) satisfy all the constraints of model (7)–(14) and, therefore, represent feasible solutions. Therefore, the optimal solution to model (7)–(14) exists. In general, linear programs are straightforward and can be solved in very little computational time using a readily available software package such as LINDO.

**Remark 5:** When the semantic of linguistic terms in \( S \) is defined by symmetrical trapezoidal fuzzy numbers, we suggest setting the initial values \( b_\alpha = CCV(s_\alpha) \). In this situation, the proposed model can reduce to the Wang and Hao model under the condition that the parameter \( \lambda = 0 \).

### C. Properties of the Proposed Model

In this section, we introduce some desired properties for the proposed model.

**Lemma 1:** For any real numbers \( x_1, x_2, y_1, y_2, |x_1 - y_1| + |x_2 - y_2| \leq |x_1 - y_1| + |x_1 - y_2|, \) if \( x_1 \leq x_2 \) and \( y_1 \leq y_2 \).

**Proof:** We distinguish six cases.

**Case A:** \( y_2 > x_2 \), and \( y_1 > x_1 \). In this case, we have \( |x_1 - y_1| + |x_2 - y_2| \geq |x_1 - y_1| + |x_1 - y_2| = 0 \).

**Case B:** \( y_2 > x_2 \), and \( x_1 \leq y_1 \leq x_2 \). In this case, we have \( |x_1 - y_1| + |x_2 - y_2| - |x_2 - y_1| - |x_1 - y_2| = -2(|x_2 - y_2| - y_1) \leq 0 \).

**Case C:** \( y_2 > x_2 \), and \( y_1 < x_1 \). In this case, we have \( |x_1 - y_1| + |x_2 - y_2| - |x_2 - y_1| - |x_1 - y_2| = -2(|x_2 - y_1| - y_1) \leq 0 \).

**Case D:** \( x_1 \leq y_1 \leq x_2 \), and \( x_1 \leq y_2 \leq x_2 \). In this case, we have \( |x_1 - y_1| + |x_2 - y_2| - |x_2 - y_1| - |x_1 - y_2| = -2(|y_2 - y_1| - y_2) \leq 0 \).

**Case E:** \( x_1 \leq y_2 \leq x_2 \), and \( y_1 < x_1 \). In this case, we have \( |x_1 - y_1| + |x_2 - y_2| - |x_2 - y_1| - |x_1 - y_2| = -2(|y_2 - y_1| - y_1) \leq 0 \).

**Case F:** \( y_2 < x_1 \). In this case, \( |x_1 - y_1| + |x_2 - y_2| - |x_2 - y_1| - |x_1 - y_2| = 0 \).

Summarizing, we have completed this proof.

**Proposition 6:** Let \( NS \) be the numerical scale derived from \( E \) using the proposed model. If \( E \) is transitive, then \( NS \) is weak ordered.

**Proof:** Using reduction to absurdity, we assume that there exist \( s_\tau, s_{\tau+1} \in S \) such that \( NS(s_\tau) > NS(s_{\tau+1}) \). Let \( NS^* = \{ NS^*(s_0), \ldots, NS^*(s_g) \} \), where

\[
NS^*(s_i) = \begin{cases} NS(s_i) & s_i \neq s_\tau, s_{\tau+1} \\ NS(s_\tau) & i = \tau + 1 \\ NS(s_{\tau+1}) & i = \tau. \end{cases}
\]

Let \( z_1 \) and \( z_2 \) be as before, and let \( f(NS) = \lambda z_1 + (1 - \lambda) z_2 \). Since

\[
f(NS) - f(NS^*) = \frac{\lambda}{\lambda} F_1 + \frac{\lambda}{\lambda} F_2 + \frac{\lambda}{\lambda} F_3 + \frac{\lambda}{\lambda} g + 1 F_4
\]

where

\[
F_1 = |NS(s_\tau) + NS(s_{\tau+1}) - NS(e_{\tau+1, \tau+1})| + |NS(s_{\tau+1}) + NS(s_{\tau+1}) - NS(e_{\tau+2, \tau+2})| - |NS(s_{\tau+1}) + NS(s_{\tau+1}) - NS(e_{\tau+1, \tau+1})| - |NS(s_\tau) + NS(s_{\tau+1}) - NS(e_{\tau+2, \tau+2})|
\]

\[
F_2 = \sum_{k=0; k \neq \tau, \tau+1}^g [NS(s_k) + NS(s_\tau) - NS(e_{\tau+1,k+1})] + |NS(s_{\tau+1}) + NS(s_k) - NS(e_{\tau+2,k+1})| - |NS(s_{\tau+1}) + NS(s_k) - NS(e_{\tau+1,k+1})| - |NS(s_\tau) + NS(s_k) - NS(e_{\tau+2,k+1})|
\]

\[
F_3 = \sum_{k=0; k \neq \tau, \tau+1}^g [NS(s_\tau) + NS(s_{\tau+1}) - NS(e_{\tau+1,\tau+1})] + |NS(s_k) + NS(s_{\tau+1}) - NS(e_{\tau+1,\tau+2})| - |NS(s_k) + NS(s_{\tau+1}) - NS(e_{\tau+1,\tau+1})| - |NS(s_k) + NS(s_{\tau+1}) - NS(e_{\tau+1,\tau+2})|
\]

\[
F_4 = |NS(s_\tau) - b_\tau| + |NS(s_{\tau+1}) - b_{\tau+1}| - |NS(s_{\tau+1}) - b_\tau| - |NS(s_\tau) - b_{\tau+1}|
\]

Since \( E \) is transitive, we have \( e_{\tau+2, \tau+2} \geq e_{\tau+1, \tau+1}, e_{\tau+2,k+1} \geq e_{\tau+1,k+1}, e_{k+1, \tau+2} \geq e_{k+1, \tau+1}, and b_{\tau+1} \geq b_{\tau} \). According to Lemma 1, we have \( F_1, F_2, F_3, F_4 \geq 0 \). Therefore, we obtain that \( f(NS) \geq f(NS^*) \), which contradicts that \( f(NS) = \min_{NS} f(NS) \). As a result, \( NS(s_\tau) \leq NS(s_{\tau+1}) \). This completes the proof of Proposition 6.

**Proposition 6** guarantees that the numerical scale \( NS \), which is derived from \( E \) using the proposed method, is weak ordered when \( E \) is transitive.

**Proposition 7:** Let \( NS^*(s_i) = b_i, (i = 0, 1, \ldots, g) \). Let \( NS \) be the numerical scale derived from \( E \) using the proposed method. Then

\[
CI(E, NS) \leq CI(E, NS^*)
\]

**Proof:** Letting \( \Omega \) be the feasible set corresponding to model (7)–(14), we have \( NS^* \in \Omega \). Therefore
\[ \lambda \text{CI}(E, NS') = \lambda \text{CI}(E, NS) + \frac{1 - \lambda}{g+1} \sum_{a=0}^{g} |NS'(s_a) - b_a| \]
\[ \geq \min_{NS \in \Omega} \{ \lambda \text{CI}(E, NS) + \frac{1 - \lambda}{g+1} \sum_{a=0}^{g} |NS(s_a) - b_a| \}. \]

Moreover, since \( NS \) is the numerical scale derived from \( E \) using the proposed method, that is, \( NS \) is the optimal solution to model (7)-(14), we have
\[ \min_{NS \in \Omega} \{ \lambda \text{CI}(E, NS) + \frac{1 - \lambda}{g+1} \sum_{a=0}^{g} |NS(s_a) - b_a| \} = \lambda \text{CI}(E, NS) + \frac{1 - \lambda}{g+1} \sum_{a=0}^{g} |NS(s_a) - b_a| \].

Consequently, \( \lambda \text{CI}(E, NS') \geq \lambda \text{CI}(E, NS) \). We obtain the proof of Proposition 7.

Proposition 7 guarantees that the transitive calibration matrix \( E \) has a better consistent index under the derived numerical scale \( NS \) than the initial one.

V. DISCUSSION ON COMPUTING NUMERICAL SCALE

A. Constructing Transitive Calibration Matrices From Linguistic Preference Relations

A natural question is how to obtain the transitive calibration matrix in decision-making problems. Let \( \{A_1, A_2, \ldots, A_n\} \) be a finite set of alternatives. When a decision maker makes pairwise comparisons using the linguistic term set, he/she can construct a linguistic preference relation to represent his/her own opinion on \( \{A_1, A_2, \ldots, A_n\} \). The linguistic preference relation based on the 2-tuple linguistic model can be formally defined as follows [1].

**Definition 14:** A discrete linguistic preference relation \( L \) on a set of alternatives \( \{A_1, A_2, \ldots, A_n\} \) is a set of linguistic terms on the product set \( X \times X \), i.e., it is characterized by a membership function
\[ u_L : X \times X \to S \]
\[ \text{Definition 15:} \text{ A continuous linguistic preference relation } L \text{ on a set of alternatives } \{A_1, A_2, \ldots, A_n\} \text{ is a set of 2-tuples on the product set } X \times X \text{, i.e., it is characterized by a membership function} \]
\[ u_L : X \times X \to S \times [-0.5, 0.5] \]
\[ \text{Remark 6: Generally, the linguistic preference relations constructed by decision makers are discrete, and the continuous linguistic preference relations can only appear in operations. When some of the elements in } L \text{ cannot be given by the decision maker, which we denote by null, we call } L \text{ the incomplete linguistic preference relation. For notation simplicity, in this paper, incomplete linguistic preference relation will also be called linguistic preference relation.} \]

Here, we propose an algorithm to construct a transitive calibration matrix from a discrete linguistic preference relation. Let \( L = (l_{ij})_{n \times n} \), where \( l_{ij} \in S \), be a discrete linguistic preference relation. For any \( l_{ik}, l_{kj} \neq l_{ij} \) null, the decision maker considers the following:
1. The preference intensity between \( A_i \) and \( A_k \) is \( l_{ik} \).
2. The preference intensity between \( A_k \) and \( A_j \) is \( l_{kj} \).
3. The preference intensity between \( A_i \) and \( A_j \) is \( l_{ij} \).

Therefore, we argue that the result of transitive calibration between \( l_{ik} \) and \( l_{kj} \) is \( l_{ij} \), that is
\[ l_{ik} \equiv l_{kj} = l_{ij}, \quad i, j, k = 1, 2, \ldots, n. \] (15)

The main idea for constructing the transitive calibration matrix from \( L \) is to detect the result of transitive calibration between \( s_a \) and \( s_b \) from (15). Let \( \Omega_{\alpha\beta} = \{p(l_{ij}) \mid l_{ik} = s_a; l_{kj} = s_b; l_{ij} \neq \text{null}; i, j, k \in \{1, 2, \ldots, n\} \} \) be the set of the position index of the result of transitive calibration between \( s_a \) and \( s_b \) provided in (15). For example, if \( l_{12} = s_1, l_{24} = s_3 \) and \( l_{14} = s_2 \), then \( 2 \in \Omega_{13} \). Based on \( \Omega_{\alpha\beta} \), we can construct the transitive calibration matrix of \( S, E = (e_{ij})_{(n+1) \times (n+1)} \). Letting \( \sharp(\Omega_{\alpha\beta}) \) be the cardinality of \( \Omega_{\alpha\beta} \), we consider four cases.

*Case A:* \( \sharp(\Omega_{\alpha\beta}) = 0 \). In this case, let \( e_{\alpha+1,\beta+1} = 0 \).
*Case B:* \( \sharp(\Omega_{\alpha\beta}) = 1 \). In this case, if \( \gamma \in \Omega_{\alpha\beta} \), then \( e_{\alpha+1,\beta+1} = s_\gamma \).
*Case C:* \( \sharp(\Omega_{\alpha\beta}) \) is an odd number and greater than 1. In this case, if \( \gamma \) is the median of the elements in \( \Omega_{\alpha\beta} \), then \( e_{\alpha+1,\beta+1} = s_\gamma \).
*Case D:* \( \sharp(\Omega_{\alpha\beta}) \) is an even number and greater than 1. In this case, if \( \gamma \) is the \( \frac{\sharp(\Omega_{\alpha\beta})}{2} \)th largest in the elements of \( \Omega_{\alpha\beta} \), then \( e_{\alpha+1,\beta+1} = (s_\gamma - 0.5) \).

The algorithm to construct the transitive calibration matrix from the discrete linguistic preference relation is listed in Table I.

B. Relationship Between Two Consistent Indexes

1) The Consistent Index of Linguistic Preference Relations Under Numerical Scales: We first generalize the concept of consistent linguistic preference relations under numerical scales.

**Definition 16:** Let \( NS \) be a numerical scale of \( S \) and \( s' \) be the midpoint of \( S \). Let \( L = (l_{ij})_{n \times n}, \) where \( l_{ij} \in S \), be a linguistic preference relation. We define the CI of \( L \) under \( NS \)
\[ \text{CI}(L, NS) = \frac{1}{n(n-1)(n-2)} \sum_{i,k=1,i \neq k, l_{ik} \neq \text{null}}^{n} \sum_{j=1, j \neq i,k}^{n} |NS(l_{ij}) \]
\[ + NS(l_{ik}) - NS(l_{ik}) - NS(s')|. \]

If CI\((L, NS) = 0\), then \( L \) is a consistent linguistic preference relation under \( NS \).

The smaller the value of CI\((L, NS)\), the more consistent \( L \) is under \( NS \). As for the Herrera and Martínez model, Alonso et al. [1, p. 10, Line 14–16] also defined the consistency index.
Table I: Algorithm for Constructing the Transitive Calibration Matrix From the Discrete Linguistic Preference Relation

<table>
<thead>
<tr>
<th>INPUT:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S = {s_0, s_1, ..., s_g}; L = (l_{ij})<em>{n \times n}$, where $l</em>{ij} \in S$</td>
</tr>
<tr>
<td>OUTPUT:</td>
</tr>
<tr>
<td>$E = (e_{ij})_{n \times n}$</td>
</tr>
<tr>
<td>BEGIN:</td>
</tr>
<tr>
<td>FOR each element $i, j \in {1, 2, ..., g + 1}$</td>
</tr>
<tr>
<td>Do $E(i, j) \leftarrow null$</td>
</tr>
<tr>
<td>$\Omega_{i-1,j-1} \leftarrow \emptyset$</td>
</tr>
<tr>
<td>END</td>
</tr>
<tr>
<td>FOR each element $i, j, k \in {1, 2, ..., n}$</td>
</tr>
<tr>
<td>Do $\alpha \leftarrow p(l_{ik})$</td>
</tr>
<tr>
<td>$\beta \leftarrow p(l_{jk})$</td>
</tr>
<tr>
<td>$\Omega_{i,j} \leftarrow \Omega_{i,j} \cup p(l_{ij})$</td>
</tr>
<tr>
<td>END</td>
</tr>
<tr>
<td>FOR each element $i, j \in {1, 2, ..., g + 1}$</td>
</tr>
<tr>
<td>Let $#(\Omega_{-1,-1})$ be the cardinality of $\Omega_{-1,-1}$</td>
</tr>
<tr>
<td>SWITCH $#(\Omega_{-1,-1})$</td>
</tr>
<tr>
<td>CASE: $#(\Omega_{-1,-1}) = 0$</td>
</tr>
<tr>
<td>$e_{ij} \leftarrow null$</td>
</tr>
<tr>
<td>CASE: $#(\Omega_{-1,-1}) = 1$</td>
</tr>
<tr>
<td>$e_{ij} \leftarrow s_\gamma$, where $\gamma \in \Omega_{-1,-1}$</td>
</tr>
<tr>
<td>CASE: $#(\Omega_{-1,-1})$ is odd number, and greater 1</td>
</tr>
<tr>
<td>$e_{ij} \leftarrow s_\gamma$, where $\gamma$ is the median in the elements of $\Omega_{-1,-1}$</td>
</tr>
<tr>
<td>CASE: $#(\Omega_{-1,-1})$ is even number, and greater 1</td>
</tr>
<tr>
<td>$e_{ij} \leftarrow (s_\gamma - 0.5)$, where $\gamma$ is the $\frac{#(\Omega_{-1,-1}) + 2}{2} th$ largest in the elements of $\Omega_{-1,-1}$</td>
</tr>
<tr>
<td>END</td>
</tr>
</tbody>
</table>

Proposition 8: When $L$ is complete linguistic preference relation, we have $CI(L, \Delta^{-1}) = CL(L)$.

Proof: In the Herrera and Martínez model, $\Delta^{-1}(s^*) = g/2$.

Remark 7: Proposition 8 shows that Definition 16 is a generalization of the consistent index of the linguistic preference relation, presented in Alonso et al. [1]. Because the Alonso et al. consistent index is based on the use of the additive transitivity property, the proposed consistency index of the linguistic preference relation has its origin in the additive transitivity property, introduced by Herrera-Viedma et al. [25], [26] and Tanino [40]. It is well known that additive transitivity property for $[0, 1]$-valued preference relations does not extend the transitivity property of crisp preference relations. Indeed, additive transitivity property for $[0, 1]$-valued preference relations is in conflict with the $[0, 1]$ scale used to provide the preference values. In [9], the consistency of reciprocal $[0, 1]$-valued preference relations is modeled via a functional equation, and it is shown that when such a function is almost continuous and monotonic (increasing), then it must be a representable uninorm. Consistency when represented by the conjunctive representable cross ratio uninorm is equivalent to Tanino’s multiplicativity transitivity property [40]. In this case, there is no conflict between the multiplicative consistency property and the $[0, 1]$ scale used to provide the preference values.

2) Linkage of Two Consistent Indexes: In Section IV-A, we introduce the consistency index of the transitive calibration matrix. In Section V-B1, we generalize the consistent index of the linguistic preference relation. In the following, we will construct a good linkage of these two consistency indexes.

For any non-null entry $e_{\alpha\beta}$ in the transitive calibration matrix $E$, we have $s_{n-1} \oplus s_{n-1} = e_{\alpha\beta}$. This means that the decision maker considers that the preference intensity between the alternatives $A_1$ and $A_2$ is $s_{n-1}$, the intensity between $A_2$ and $A_3$ is $s_{n-1}$, and the intensity between $A_1$ and $A_3$ is $e_{\alpha\beta}$. Thus, according to $s_{n-1} \oplus s_{n-1} = e_{\alpha\beta}$, we can construct a $3 \times 3$ linguistic preference relation to represent the decision maker’s opinion on $\{A_1, A_2, A_3\}$, that is,

$$T(e_{\alpha\beta}) = (t_{ij}^{(e_{\alpha\beta})})_{3 \times 3} = \begin{bmatrix} \text{null} & s_{n-1} & e_{\alpha\beta} \\ \text{null} & s_{n-1} & e_{\alpha\beta} \\ \text{null} & \text{null} & \text{null} \end{bmatrix}$$

where null denotes undefined entries. Thus, in essence, we argue that $CI(T(e_{\alpha\beta}), NS)$ reflects the consistent degree of the transitive calibration $s_{n-1} \oplus s_{n-1} = e_{\alpha\beta}$.

**Proposition 9:** $CI(E, NS) = \frac{1}{m} \sum_{\beta=1}^{g+1} \sum_{\alpha,\beta \neq \text{null}} \sum_{\alpha=1}^{g+1} CI(T(e_{\alpha\beta}), NS)$, where $m$ is the number of the non-null elements of $E$.

**Proof:** According to Definitions 12 and 16, we have Proposition 9.

**Proposition 10:** Let $E$ be the transitive calibration matrix constructed from the linguistic preference relation $L = (l_{ij})_{n \times n}$. Let $\#(\Omega_{\alpha\beta})$ and $T(e_{\alpha\beta})$ be as before. When $\#(\Omega_{\alpha\beta}) < 2$, for $\alpha, \beta = 0, 1, \ldots, g$, we have

$$CI(L, NS) = \frac{1}{n(n-1)(n-2)} \sum_{i,k=1}^{n} \sum_{i \neq k} \sum_{j=1}^{n} CI(T(e_{p(ij),p(l_{ik})}), NS).$$

**Proof:** According to the algorithm for constructing $E$ from $L$, we have $e_{p(ij),p(l_{ik})} = l_{ik}$ when $\#(\Omega_{\alpha\beta}) < 2$. Thus

$$CI(T(e_{p(ij),p(l_{ik})}), NS) = |NS(l_{ij}) + NS(l_{ik}) - NS(l_{ik}) - NS(s^*)|$$

and

$$CI(L, NS) = \frac{1}{n(n-1)(n-2)} \sum_{i,k=1}^{n} \sum_{i \neq k} \sum_{j=1}^{n} CI(T(e_{p(ij),p(l_{ik})}), NS).$$

This completes the proof of Proposition 10.

**Note 2:** $\#(\Omega_{\alpha\beta})$ is the number of the result of transitive calibration between $s_{\alpha}$ and $s_{\beta}$ provided in (15). In more cases, we argue that the decision maker is rational in constructing linguistic preference relation, and therefore, the result of transitive calibration between $s_{\alpha}$ and $s_{\beta}$, provided in (15), is sole or null for more cases, that is, $\#(\Omega_{\alpha\beta}) < 2$.

**Remark 8:** Proposition 9 shows that the consistent index of the transitive calibration matrix $E$ has a definite
Corollary 3: Let \( E \) be the transitive calibration matrix constructed from the linguistic preference relation \( L = (l_{ij})_{n \times n} \). If \( L \) is consistent under NS, then \( E \) is consistent under NS.

Proof: According to Definitions 13 and 16 and Propositions 9 and 10, we have Corollary 3.

Corollary 4: Let \( E \) be the transitive calibration matrix constructed from \( L = (l_{ij})_{n \times n} \). If \( L \) is consistent under NS, then \( E \) is symmetrical and transitive.

Proof: According to Proposition 5 and Corollary 3, we have Corollary 4.

Remark 9: Corollaries 3 and 4 guarantee that the transitive calibration matrix, constructed from consistent linguistic preference relation, is consistent, symmetrical, and transitive. However, it is hard to obtain consistent linguistic preference relations, especially when the number of alternatives is large. In the future research, we suggest proposing methods to deal with inconsistency in linguistic preference relations under the concept of numerical scale.

C. Summary

In previous sections, we define the concept of numerical scale and extend the 2-tuple fuzzy linguistic representation models under numerical scale. By developing the concept of the transitive calibration matrix, we propose an optimization model to help the decision maker compute the numerical scale of the linguistic term set. The implementation of computing numerical scale requires the following two-step procedure (see, Fig. 1).

Step 1: Obtain the transitive calibration matrix of the linguistic term set. In general, we obtain the transitive calibration matrix by asking the decision maker to provide the answer of the transitive calibration between two linguistic terms. For decision problems using linguistic preference relations, we suggest using the algorithm introduced in Section V-A to construct the transitive calibration matrix.

Step 2: Compute the numerical scale from the transitive calibration matrix. The proposed model is a two-objective constrained optimization model. By using the linear weighted summation method, and introducing transformed decision variables, we can transform it into a linear programming model.

Comparing the existing models (i.e., the Herrera and Martínez model and the Wang and Hao model), our proposal has the following characteristics.

1) Comparing the Herrera and Martínez model, the proposed model can deal with linguistic variables with linguistic term sets that are not uniformly and symmetrically distributed.
2) The Wang and Hao model obtains the numerical scale (i.e., CCVs) from the semantic information of linguistic terms defined by fuzzy numbers and has some limitations mentioned in Section I. Different from the Wang and Hao model, our model derives the numerical scale from the transitive calibration matrix and may make the transitive calibration information of the decision maker more consistent.
3) Different decision makers who may have different background and knowledge often need to set different numerical scales for the same linguistic term set. Our model can set suitable numerical scales for different decision makers.

VI. ILLUSTRATIVE EXAMPLE

In order to show that how these theoretical results work in practice, let us consider two examples.

A. Example 1

In Example 1, we consider the linguistic term set

\[ S_{Example1} = \{s_0 = I, s_1 = EU, s_2 = VLC, s_3 = SC, s_4 = IM, s_5 = MC, s_6 = ML, s_7 = EL, s_8 = C\} \]

used by Wang and Hao [42]. In \( S_{Example1} \), the midterm \( s^* = s_4 \). \( S_{Example1} \) is defined by symmetrical trapezoidal fuzzy numbers in \([0, 1]\) as follows:

\[
\begin{align*}
I & \quad \text{Impossible} \quad T[0, 0, 0, 0], \\
EU & \quad \text{Extremely unlikely} \quad T[0, 0.01, 0.03, 0.04], \\
VLC & \quad \text{Very low chance} \quad T[0.03, 0.10, 0.18, 0.25], \\
SC & \quad \text{Small chance} \quad T[0.18, 0.22, 0.36, 0.40], \\
IM & \quad \text{It may} \quad T[0.36, 0.42, 0.58, 0.64], \\
MC & \quad \text{Meanful chance} \quad T[0.60, 0.64, 0.78, 0.82], \\
ML & \quad \text{Most likely} \quad T[0.75, 0.82, 0.90, 0.97], \\
EL & \quad \text{Extremely likely} \quad T[0.96, 0.97, 0.99, 1], \\
C & \quad \text{Certain} \quad T[1, 1, 1, 1]. 
\end{align*}
\]
Using the Wang and Hao model, we obtain the values of the CCVs, which are listed in Table II. In the following, we compute the numerical scale of \( S_{\text{Example1}} \) using the proposed model.

1) Construct the transitive calibration matrix \( E \) of \( S_{\text{Example1}} \):

In this example, the decision maker constructs \( E \), by directly providing the answers of transitive calibration between \( s_1 \) and \( s_2 \), \( \alpha, \beta = 0, 1, \ldots , 8 \). \( E \) is listed, as shown at the bottom of the page, where null denotes undefined entries. According to Definition 11, we know that \( E \) is transitive.

2) Compute the numerical scale of \( S_{\text{Example1}} \) by solving linear programming model: Let initial values provided by the decision maker be \( b_{\alpha} = \text{CCV}(s_{\alpha}) \) \( (\alpha = 0, 1, \ldots , 8) \). Let the parameter \( \lambda \) used in linear programming model (7)–(14), equal 0.5. Then, solving model (7)–(14), we obtain the numerical scale \( \text{NS}(s_{\alpha}) \), \( i = 0, 1, \ldots , 8 \). The values of \( \text{NS}(s_{\alpha}) \) are listed in Table II.

We find that the derived numerical scale \( \text{NS} \) is ordered. Let \( \text{NS}'(s_{\alpha}) = \text{CCV}(s_{\alpha}) \) \( (\alpha = 0, 1, \ldots , 8) \), we have \( \text{CI}(E, \text{NS}) = 0.0614 \leq \text{CI}(E, \text{NS}') = 0.1177 \). Furthermore, according to \( s_{\alpha-1} \oplus s_{\beta-1} = e_{\alpha \beta} \) \( (\alpha, \beta = 0, 1, \ldots , 8) \), we construct 3 \( \times \) 3 linguistic preference relations to represent the decision maker’s opinion, which are \( T^{(e_{\alpha \beta})} \) \( (\alpha, \beta = 0, 1, \ldots , 8) \). Let \( C = (c_{ij})_{9 \times 9} \), where \( c_{ij} = \text{CI}(T^{(e_{ij})}, \text{NS}') \), and \( C' = (c_{ij}')_{9 \times 9} \), where \( c_{ij}' = \text{CI}(T^{(e_{ij}')}, \text{NS}') \). \( C \) and \( C' \) are listed, as shown at the bottom of the next page. From \( C \) and \( C' \), we find that the percent of \( \text{CI}(T^{(e_{ij})}, \text{NS}') \leq \text{CI}(T^{(e_{ij}')}, \text{NS}') \) is 88.9.

### Table II

VALUES OF THE CCV \( s_{\alpha} \) AND \( \text{NS}(s_{\alpha}) \)

<table>
<thead>
<tr>
<th>( s_{\alpha} )</th>
<th>( \alpha = 0 )</th>
<th>( \alpha = 1 )</th>
<th>( \alpha = 2 )</th>
<th>( \alpha = 3 )</th>
<th>( \alpha = 4 )</th>
<th>( \alpha = 5 )</th>
<th>( \alpha = 6 )</th>
<th>( \alpha = 7 )</th>
<th>( \alpha = 8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{CCV}(s_{\alpha}) )</td>
<td>0</td>
<td>0.02</td>
<td>0.14</td>
<td>0.29</td>
<td>0.5</td>
<td>0.71</td>
<td>0.86</td>
<td>0.98</td>
<td>1</td>
</tr>
<tr>
<td>( \text{NS}(s_{\alpha}) )</td>
<td>0</td>
<td>0.02</td>
<td>0.22</td>
<td>0.35</td>
<td>0.5</td>
<td>0.63</td>
<td>0.79</td>
<td>0.91</td>
<td>1</td>
</tr>
</tbody>
</table>

In \( S_{\text{Example2}} \), the midterm \( s^* = s_2 \). Based on \( S_{\text{Example2}} \), a decision maker supplies a linguistic preference relation \( L \)

\[
L = \begin{pmatrix}
    s_2 & s_3 & s_1 & s_4 & s_0 \\
    \text{null} & s_0 & s_0 & s_0 & s_0 \\
    \text{null} & \text{null} & s_2 & s_4 & s_1 \\
    \text{null} & \text{null} & \text{null} & s_2 & \text{null} \\
    \text{null} & \text{null} & \text{null} & \text{null} & s_2
\end{pmatrix}
\]

where null denotes undefined entries.

Here, we use the proposed model to compute the numerical scale of \( S_{\text{Example2}} \).

1) Construct the transitive calibration matrix \( F \) from \( L \): In this example, the decision maker constructs \( F \) from \( L \), using the algorithm introduced in Section V-A. Let \( \Omega_{\alpha \beta} = \{ p(l_{ij}) \mid l_{ij} = s_{\alpha} \wedge l_{ij} \in \{ s_2 \wedge i, j \in \{ 1, 2, \ldots , 5 \} \} \). \( \Omega_{\alpha \beta} \) \( (\alpha, \beta = 0, 1, \ldots , 4) \)

are listed as follows:

\[
\begin{align*}
\Omega_{0,0} & = \phi, \quad \Omega_{0,1} = \{ 0 \}, \quad \Omega_{1,0} = \phi, \quad \Omega_{0,2} = \{ 0 \} \\
\Omega_{2,0} & = \{ 0 \}, \quad \Omega_{0,3} = \phi, \quad \Omega_{3,0} = \{ 1, 0 \}, \quad \Omega_{0,4} = \{ 3 \} \\
\Omega_{4,0} & = \phi, \quad \Omega_{1,1} = \{ 0 \}, \quad \Omega_{1,2} = \{ 1 \}, \quad \Omega_{2,1} = \{ 1 \} \\
\Omega_{1,3} & = \phi, \quad \Omega_{3,1} = \phi, \quad \Omega_{1,4} = \{ 4 \}, \quad \Omega_{4,1} = \phi \\
\Omega_{2,2} & = \{ 2 \}, \quad \Omega_{2,3} = \{ 3 \}, \quad \Omega_{3,2} = \{ 3 \}, \quad \Omega_{2,4} = \{ 4 \} \\
\Omega_{4,2} & = \{ 4 \}, \quad \Omega_{3,3} = \{ 4 \}, \quad \Omega_{3,4} = \phi, \quad \Omega_{4,3} = \phi, \quad \Omega_{4,4} = \phi.
\end{align*}
\]

According to \( \Omega_{\alpha \beta} \) \( (\alpha, \beta = 0, 1, \ldots , 4) \) and \( \frac{s}{2}(\Omega_{\alpha \beta}) \) we construct the transitive calibration matrix \( F \) of \( S_{\text{Example2}} \)

\[
F = \begin{pmatrix}
    \text{null} & s_0 & s_0 & \text{null} & s_3 \\
    \text{null} & s_0 & s_1 & \text{null} & s_4 \\
    s_0 & s_1 & s_2 & s_3 & s_4 \\
(0.5s_2, 0.5s_3) & \text{null} & s_4 & \text{null} & \text{null} \\
\end{pmatrix}
\]
where null denotes undefined entries. According to Definition 11, we find that $F$ is transitive.

2) Compute the numerical scale of $S^{\text{Example}2}$ by solving linear programming model: Let initial values provided by the decision maker be $b = \{0, 0.2, 0.5, 0.8, 1\}$. Let the parameter $\lambda$, used in linear programming model (7)–(14), equal 0.5. Then, using model (7)–(14), we obtain the numerical scale $NS(s_\alpha), \alpha = 0, 1, \ldots, 4$. The values of $NS(s_\alpha), \alpha = 0, 1, \ldots, 4$ are listed in Table III.

We find that the derived numerical scale $NS$ is ordered. Letting $NS'(s_\alpha) = b_\alpha, (\alpha = 0, 1, \ldots, 4)$, we have $CI(E, NS) = 0.064 \leq CI(E, NS') = 0.087$. Furthermore, according to $s_{\alpha-1} \oplus s_{\beta-1} = f_{\alpha, \beta} (\alpha, \beta = 0, 1, \ldots, 4)$, we construct $3 \times 3$ linguistic preference relations to represent the decision maker’s opinion, which are $T'(f_{s_\alpha}, s_{\alpha}) (\alpha, \beta = 0, 1, \ldots, 4)$. Let $D = (d_{ij})_{3\times5}$, where $d_{ij} = CI(T(f_{s_\alpha}), NS)$, and $D' = (d'_{ij})_{3\times5}$, where $d'_{ij} = CI(T'(f_{s_\alpha}), NS')$. $D$ and $D'$ are listed as

$$
D = \begin{pmatrix}
\text{null} & 0.26 & 0 & \text{null} & 0.26 \\
\text{null} & 0.02 & 0 & \text{null} & 0.26 \\
0 & 0 & 0 & 0 & 0 \\
0.38 & \text{null} & 0 & 0.02 & \text{null} \\
\text{null} & \text{null} & \text{null} & \text{null} & \text{null}
\end{pmatrix}
$$

$$
D' = \begin{pmatrix}
\text{null} & 0.3 & 0 & \text{null} & 0.3 \\
\text{null} & 0.1 & 0 & \text{null} & 0.3 \\
0 & 0 & 0 & 0 & 0 \\
0.4 & \text{null} & 0.1 & \text{null} & \text{null} \\
\text{null} & \text{null} & \text{null} & \text{null} & \text{null}
\end{pmatrix}.
$$

From $D$ and $D'$, we find that $CI(T'(f_{s_\alpha}), NS) \leq CI(T'(f_{s_\alpha}), NS')$ for $\alpha, \beta = 0, 1, \ldots, 4$.

VII. CONCLUSION

Herrera and Martínez [22]–[24] initiated the study of the interesting 2-tuple fuzzy linguistic representation model. However, Herrera and Martínez have pointed out [22, Conclusion] that this approach is only suitable for linguistic term sets with a uniform and symmetrical distribution. Wang and Hao [42, Conclusion] provided a generalized version of the 2-tuple fuzzy linguistic representation model for CW, based on the concepts of symbolic proportion and CCVs of linguistic terms. The Wang and Hao model can deal with linguistic variables with linguistic term sets that are not uniformly and symmetrically distributed. However, the semantic of linguistic terms in the linguistic term set used in the Wang and Hao model can only be defined by symmetrical trapezoidal fuzzy numbers. Thus, Wang and Hao [42, Conclusion] suggested investigating how to generalize their proposal by considering linguistic 2-tuples under more general contexts. In this paper, we propose the concepts of numerical scale and transitive calibration matrix. We extend the 2-tuple fuzzy linguistic representation model under numerical scale and develop an optimization model to compute the numerical scale of the

---

<table>
<thead>
<tr>
<th>$s_\alpha$</th>
<th>$\alpha = 0$</th>
<th>$\alpha = 1$</th>
<th>$\alpha = 2$</th>
<th>$\alpha = 3$</th>
<th>$\alpha = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$NS(s_\beta)$</td>
<td>0</td>
<td>0.24</td>
<td>0.5</td>
<td>0.76</td>
<td>1</td>
</tr>
</tbody>
</table>

---

**TABLE III**

VALUES OF $NS(s_\alpha)$
linguistic term set from the transitive calibration matrix. The major contributions and findings are as given in the following points.

1) This paper defines the concept of numerical scale and extends the 2-tuple fuzzy linguistic representation model under numerical scale. We show that our proposal reduces to the Herrera and Martínez model and the Wang and Hao model, when, respectively, setting corresponding numerical scales.

2) We propose the concept of transitive calibration matrix, and define its symmetry, transitivity, and consistent index. We develop an optimization model to compute the numerical scale of the linguistic term set from the transitive calibration matrix. The optimization model makes the transitive calibration information of the decision maker more consistent. In particular, our proposal can set suitable numerical scales for different decision makers.

3) We present an algorithm to automatically construct the transitive calibration matrix for decision problems using linguistic preference relations.

4) We extend the consistent index of the linguistic preference relation, defined in Alonso et al. [1], under the concept of numerical scale. We analyze the linkage between the extended consistent index of the linguistic preference relation and the one of the transitive calibration matrix.

Also, for linguistic term sets that are not uniformly and symmetrically distributed, Herrera et al. [20] have recently defined them unbalanced linguistic term sets. Herrera et al. [20] and Herrera-Viedma and López-Herrera [28] also developed an interesting methodology to deal with decision problems using unbalanced linguistic information. This methodology is based on the use of linguistic hierarchy [10], [24], [31] and the 2-tuple fuzzy linguistic representation model. Naturally, the proposal of this paper also can deal with the unbalanced linguistic term sets. A comparative study of these two research lines may be interesting future work.

ACKNOWLEDGMENT

The authors would like to thank the editors and the anonymous referees for their valuable comments and suggestions.

REFERENCES


Authors’ photographs and biographies not available at the time of publication.